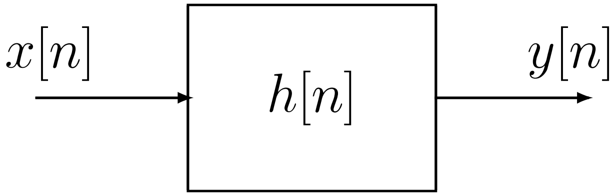
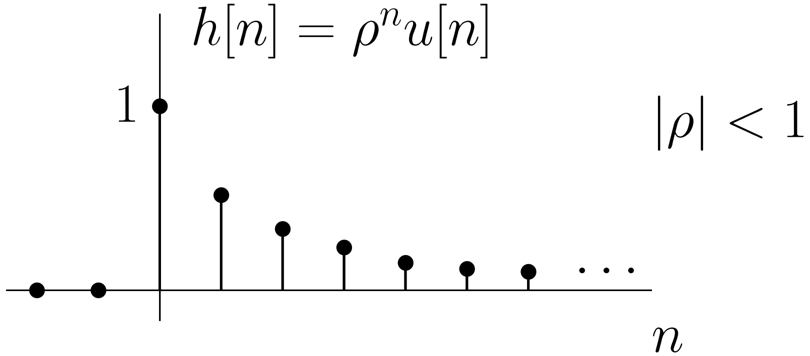
EXAMPLE 5.1

The linear shift-invariant system shown below is driven by a process with mean m_{o} and covariance function $C_x[l] = \sigma_{\mathrm{o}}^2\delta[l]$. (This is white noise with an added nonzero mean.)





It is desired to compute the mean, correlation function, and covariance function of the output, and the cross-correlation and cross-covariance functions between input and output.

The mean of the output is very simple to compute:

$$m_y = m_x \sum_{k=-\infty}^{\infty} h[k] = m_\text{o} \sum_{k=0}^{\infty} \rho^k = \frac{m_\text{o}}{1-\rho}$$

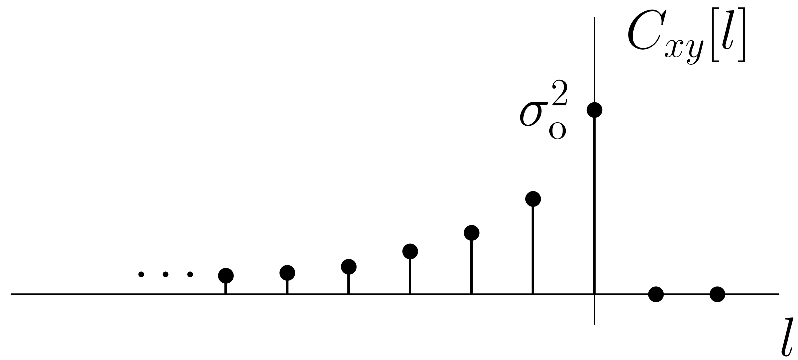
Since the input and output have nonzero mean, it is easiest to first compute the auto- and cross-*covariance* functions. Then the corresponding correlation functions can be computed by taking account of the mean.

The cross-covariance of the output is given by

$$C_{yx}[l] = h[l] * C_x[l] = \left(\rho^l u[l]\right) * \left(\sigma_\text{o}^2 \delta[l]\right) = \sigma_\text{o}^2 \rho^l u[l]$$

and therefore

$$C_{xy}[l] = C_{yx}^*[-l] = \sigma_\text{o}^2 (\rho^*)^{-l} u[-l]$$



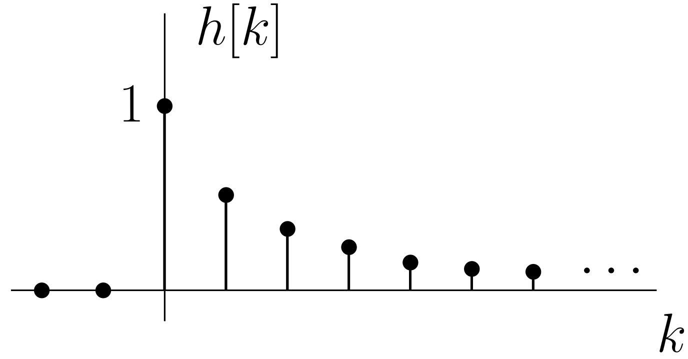
Then the autocovariance follows from

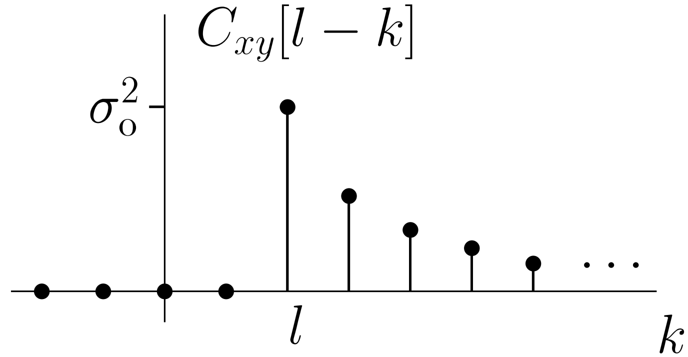
$$C_y[l] = h[l] * C_{xy}[l] = \sum_{k=-\infty}^{\infty} h[k] C_{xy}[l-k]$$

To help in carrying out the convolution, the terms in the summation are depicted below for a typical value of $l > 0$.

(continued on the next page)

$C_y[l] = \Sigma_{k=-\infty}^{\infty} h[k] C_{xy}[l-k]$





Thus for $l > 0$ the summation is

$$C_y[l] = \sum_{k=l}^{\infty} \rho^k \cdot \sigma_{\mathrm{o}}^2 (\rho^*)^{-(l-k)}$$

Upon making the substitution $i = k - l$ this becomes

$$C_y[l] = \sigma_o^2 \sum_{i=0}^{\infty} \rho^{i+l}(\rho^*)^i = \sigma_o^2 \rho^l \sum_{i=0}^{\infty} \left(|\rho|^2\right)^i = \frac{\sigma_o^2 \rho^l}{1-|\rho|^2}; \quad l > 0$$

In a similar manner, for $l \leq 0$ we find

$$C_y[l] = \frac{\sigma_o^2 (\rho^*)^{-l}}{1-|\rho|^2}; \quad l \leq 0$$

The cross-correlation function can now be computed as

$$\begin{aligned} R_{xy}[l] &= C_{xy}[l] + m_x m_y^* \\ &= \sigma_o^2 (\rho^*)^{-l} u[-l] + m_o \cdot \left(\frac{m_o}{1-\rho}\right)^* \\ &= \sigma_o^2 (\rho^*)^{-l} u[-l] + \frac{|m_o|^2}{1-\rho^*} \end{aligned}$$

and the autocorrelation function of the output is

$$\begin{aligned} R_y[l] &= C_y[l] + |m_y|^2 \\ &= \begin{cases} \frac{\sigma_o^2}{1-|\rho|^2}\rho^l + \left|\frac{m_o}{1-\rho}\right|^2 & l > 0 \\ \frac{\sigma_o^2}{1-|\rho|^2}(\rho^*)^{-l} + \left|\frac{m_o}{1-\rho}\right|^2 & l \leq 0 \end{cases} \end{aligned}$$

Observe that when the mean is zero, this is the exponential correlation function encountered before. This shows that a process with the exponential correlation function can always be generated by applying white noise to a stable first order system. The variance parameter σ^2 of the process is given by

$$\sigma^2 = \frac{\sigma_{\text{o}}^2}{1 - |\rho|^2}$$

In the real case (with zero mean) the correlation function has the simpler form

$$R_y[l] = C_y[l] = \frac{\sigma_{\text{o}}^2}{1 - \rho^2}\rho^{|l|}\,; \qquad \forall l$$

$\square$

EXAMPLE 5.5

A complex spectral density function has the form

$$S_x(z) = e^{\frac{1+z^2}{z}} = e^{z^{-1}+z}$$

This function satisfies the required condition $S_x(z) = S_x^*(1/z^*)$. The power spectral density function is

$$S_x(e^{\jmath\omega}) = e^{2\cos\omega}$$

which is positive for all values of ω. The function satisfies the Paley–Wiener condition since

$$\int_{-\pi}^{\pi} |\ln S_x(e^{\jmath\omega})| d\omega = \int_{-\pi}^{\pi} |2\cos\omega| d\omega < \infty$$

The factorization can be done by inspection to obtain

$$S_x(z) = 1 \cdot e^{z^{-1}} \cdot e^{z}$$

where the causal factor is seen to be

$$H_{ca}(z) = e^{z^{-1}}$$

which converges everywhere except at $z = 0$.

The impulse response of the filter is given by

$$h_{ca}[n] = \frac{1}{n!}u[n]$$

where $u[n]$ is the unit step function. This follows because

$$H_{ca}(z) = e^{z^{-1}} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$\square$

EXAMPLE 5.101

The complex spectral density function

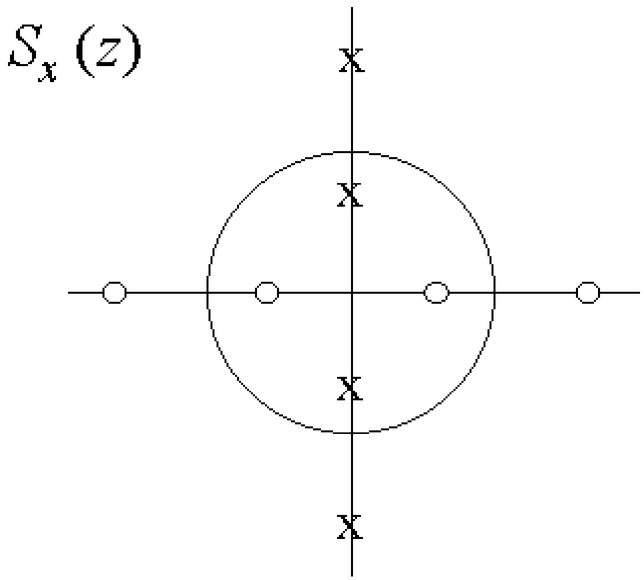
$$S_x(z) = \frac{-4z^2 + 10 - 4z^{-2}}{2z^2 + 5 + 2z^{-2}}$$

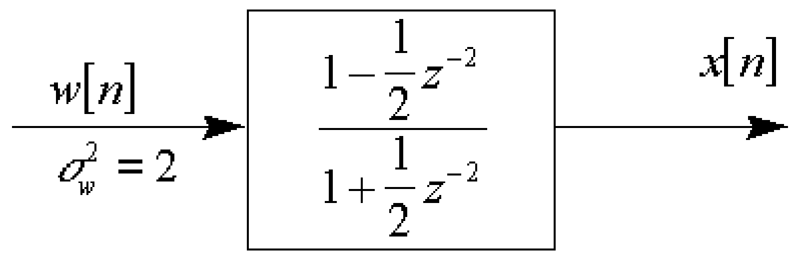
can be factored as

$$\begin{aligned}
S_x(z) &= -2\,\frac{1 - \frac{5}{2}z^{-2} + z^{-4}}{1 + \frac{5}{2}z^{-2} + z^{-4}} = -2\frac{\left(1 - \frac{1}{2}z^{-2}\right)\left(1 - 2z^{-2}\right)}{\left(1 + \frac{1}{2}z^{-2}\right)\left(\left(1 + 2z^{-2}\right)\right.} \\
&= \underset{\substack{\uparrow \\ \underline{\text{not}}\ \mathcal{K}_{\mathrm{o}}}}{-2} \cdot \underbrace{\frac{\left(1 - \frac{1}{\sqrt{2}}z^{-1}\right)\left(1 + \frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1 - \jmath\frac{1}{\sqrt{2}}z^{-1}\right)\left(1 + \jmath\frac{1}{\sqrt{2}}z^{-1}\right)}}_{H_{ca}(z)} \cdot \underbrace{\frac{\left(1 - \sqrt{2}\,z^{-1}\right)\left(1 + \sqrt{2}\,z^{-1}\right)}{\left(1 - \jmath\sqrt{2}\,z^{-1}\right)\left(1 + \jmath\sqrt{2}\,z^{-1}\right)}}_{\underline{\text{not}}\ H_{ca}(z^{-1})}
\end{aligned}$$

This expression can be rewritten as

$$S_x(z) \;=\; \underset{\mathcal{K}_\mathrm{o}}{\overset{\textstyle 2}{\uparrow}} \cdot \underbrace{\frac{\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1-\jmath\frac{1}{\sqrt{2}}z^{-1}\right)\left(1+\jmath\frac{1}{\sqrt{2}}z^{-1}\right)}}_{H_{ca}(z)} \cdot \underbrace{\frac{\left(1-\frac{1}{\sqrt{2}}\,z\right)\left(1+\frac{1}{\sqrt{2}}\,z\right)}{\left(1-\jmath\frac{1}{\sqrt{2}}\,z\right)\left(1+\jmath\frac{1}{\sqrt{2}}\,z\right)}}_{H_{ca}(z^{-1})}$$





Innovations Representation

□

EXAMPLE 5.6

A complex spectral density function for a certain real random process is

$$S_x(z) = \frac{-(1/a)}{z - (a + 1/a) + z^{-1}}$$

This can be written in the equivalent form

$$S_x(z) = \frac{1}{-az + (1 + a^2) - az^{-1}} = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - az}$$

which leads to the correct identification $\mathcal{K}_\mathrm{o} = 1$ and

$$H_{ca}(z) = \frac{1}{1 - az^{-1}}$$

Notice a possible pitfall here. Suppose the function had been factored as

$$S_x(z) = \frac{1}{-az + (1 + a^2) - az^{-1}} = \frac{1}{(z - a)(z^{-1} - a)}$$

then it might be tempting to take

$$H_{ca}(z) = \frac{1}{z - a} = \frac{z^{-1}}{1 - az^{-1}} \quad \text{(I)}$$

since it satisfies the symmetry condition

$$H_{ca}^*(1/z^*) = H_{ca}(z^{-1}) = \frac{1}{z^{-1} - a}$$

However the term (I) is *not minimum-phase.* It has a zero at $z = \infty$ for one thing. The inverse z-transform is

$$a^{n-1}u[n-1]$$

where $u[n]$ is the unit step function, so the partial energy is not smaller than that of the impulse response

$$a^n u[n]$$

which *is* minimum-phase. Also the inverse

$$H_{ca}^{-1}(z) = z - a$$

is not causal.

This problem can be resolved by supplying an extra pole and zero at the origin; that is, by writing $S_x(z)$ in the equivalent form

$$S_x(z) = \frac{z \cdot z^{-1}}{-az + (1 + a^2) - az^{-1}} = \frac{z}{(z-a)} \cdot \frac{z^{-1}}{(z^{-1} - a)}$$

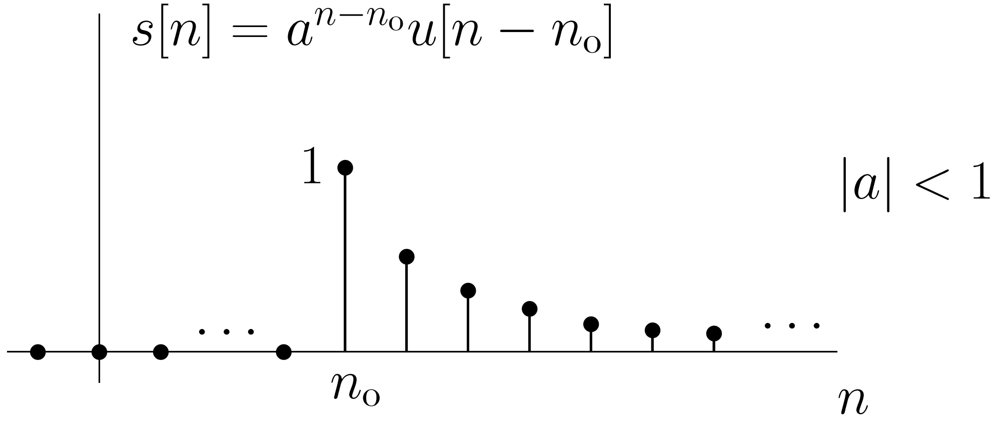
This is now a correct spectral factorization with

$$H_{ca}(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

which is truly minimum-phase. $\square$

EXAMPLE 5.4

A simple real transient signal has the form shown below



where $u[n]$ is the unit step function. It is desired to design an FIR matched filter to detect this signal in noise.

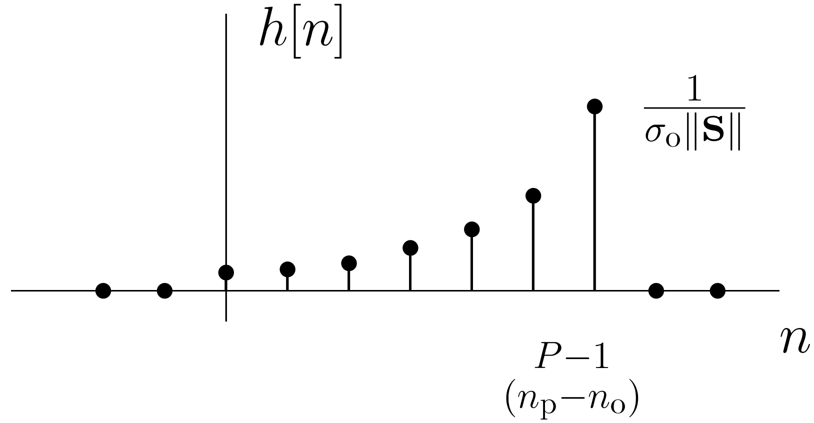
If the signal is regarded as having some finite length P, after which it is essentially zero, then the impulse response of the matched filter, is proportional to the reversed truncated signal

$$h[n] = \frac{1}{\sigma_o \|\mathbf{s}\|} \cdot a^{P-1-n} \qquad 0 \leq n \leq P-1$$

The normalizing constant is given by

$$\sigma_o \|\mathbf{s}\| = \sigma_o \left(\sum_{k=0}^{P-1} (a^k)^2 \right)^{1/2} = \sigma_o \left(\frac{1 - a^{2P}}{1 - a^2} \right)^{1/2}$$

The impulse response is depicted below.



The signal-to-noise ratio is given by

$$\text{SNR} = \frac{\|\mathbf{s}\|^2}{\sigma_o^2} = \frac{1}{\sigma_o^2} \left(\frac{1 - a^{2P}}{1 - a^2} \right)$$

Now consider the case where the noise is not white, but has the exponential correlation function.

$$R_\eta[l] = \sigma\rho^{|l|} = \frac{\sigma_o^2}{1-\rho^2}\rho^{|l|}$$

Problem 3.26 in Chapter 3 shows that the inverse correlation matrix corresponding to this correlation function has a particularly simple banded form. This is depicted below for the case of $P = 5$.

$$\mathbf{R}_{\boldsymbol{\eta}}^{-1} = \frac{1}{\sigma_o^2}\begin{bmatrix} 1 & -\rho & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$$

The matched filter thus has the form

$$\mathbf{h} = \frac{1}{\sqrt{\mathbf{s}^{*T}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\mathbf{s}}}\mathbf{R}_{\boldsymbol{\eta}}^{-1}\tilde{\mathbf{s}}^* = \frac{1}{\sqrt{\mathrm{SNR}}}\frac{1}{\sigma_o^2}\begin{bmatrix} 1 & -\rho & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & -\rho & 1 \end{bmatrix}\begin{bmatrix} a^4 \\ a^3 \\ a^2 \\ a \\ 1 \end{bmatrix}$$

where SNR here represents the maximum signal-to-noise ratio, achieved by the matched filter. The terms of the matched filter are

$$\begin{aligned} h[0] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}\left[a^4 - \rho a^3\right] \\ h[n] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}\left[(1-\rho^2)a^{4-n} - \rho a^{3-n} - \rho a^{5-n}\right] \qquad\qquad 1 \le n \le 3 \\ h[4] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}\left[1 - \rho a\right] \end{aligned}$$

To evaluate SNR, observe that the inverse correlation matrix can be factored as

$$\mathbf{R}_{\boldsymbol{\eta}}^{-1} = \frac{1}{\sigma_{\mathrm{o}}^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 & 0 \\ 0 & -\rho & 1 & 0 & 0 \\ 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & -\rho & 1 \end{bmatrix}\begin{bmatrix} 1 & -\rho & 0 & 0 & 0 \\ 0 & 1 & -\rho & 0 & 0 \\ 0 & 0 & 1 & -\rho & 0 \\ 0 & 0 & 0 & 1 & -\rho \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore SNR can be written as

$$\mathrm{SNR} = \mathbf{s}^T\mathbf{R}_{\boldsymbol{\eta}}^{-1}\mathbf{s} = \mathbf{s}^T\mathbf{R}_{\boldsymbol{\eta}}^{-1/2}(\mathbf{R}_{\boldsymbol{\eta}}^{-1/2})^T\mathbf{s} = (\mathbf{s}')^T\mathbf{s}' = \|\mathbf{s}'\|^2$$

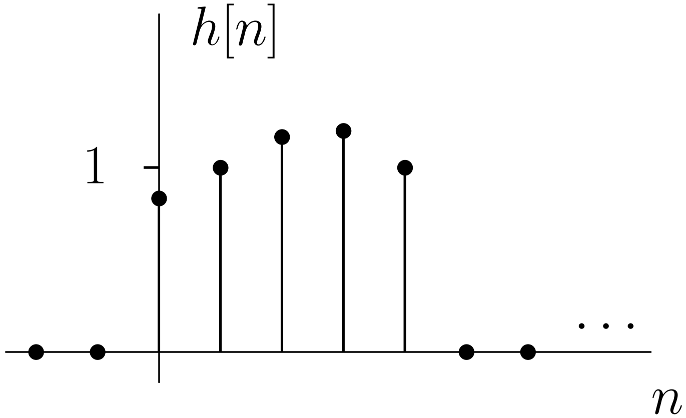
where

$$\mathbf{s}' = (\mathbf{R}_{\boldsymbol{\eta}}^{-1/2})^T \mathbf{s} = \frac{1}{\sigma_\text{o}} \begin{bmatrix} 1 & -\rho & 0 & 0 & 0 \\ 0 & 1 & -\rho & 0 & 0 \\ 0 & 0 & 1 & -\rho & 0 \\ 0 & 0 & 0 & 1 & -\rho \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \\ a^4 \end{bmatrix} = \frac{1}{\sigma_\text{o}} \begin{bmatrix} 1-\rho a \\ a(1-\rho a) \\ a^2(1-\rho a) \\ a^3(1-\rho a) \\ a^4 \end{bmatrix}$$

SNR is then given by

$$\text{SNR} = \|\mathbf{s}'\|^2 = \frac{1}{\sigma_\text{o}^2}(1-\rho a)^2 \left[1 + a^2 + a^4 + a^6 + a^8/(1-\rho a)^2\right]$$

The filter impulse response is depicted below for the parameter values $a = 0.95$, $\sigma_\text{o}^2 = 0.25$, and $\rho = -0.40$ (negatively correlated noise). Note that when the noise is not white, $h[n]$ does not necessarily resemble the signal (see next page).



SNR = 29.0

(Values are 0.86, 1.05, 1.10, 1.16, 1.03)

It is not too difficult to generalize the above formulas for the special case of $P = 5$ to an arbitrary value of P. The results are:

$$
\begin{aligned}
h[0] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}(a-\rho)a^{P-2} \\
h[n] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}\left((1-\rho^2)a-\rho(1+a^2)\right)a^{P-2-n} \qquad 1 \le n \le P-2 \\
h[P-1] &= \frac{1}{\sigma_{\mathrm{o}}^2\sqrt{\mathrm{SNR}}}(1-\rho a)
\end{aligned}
$$

where the value of SNR is

$$\mathrm{SNR} = \frac{1}{\sigma_{\mathrm{o}}^2}(1-\rho a)^2\left[\frac{1-a^{2(P-1)}}{1-a^2} + \frac{a^{2(P-1)}}{(1-\rho a)^2}\right]$$

$\square$